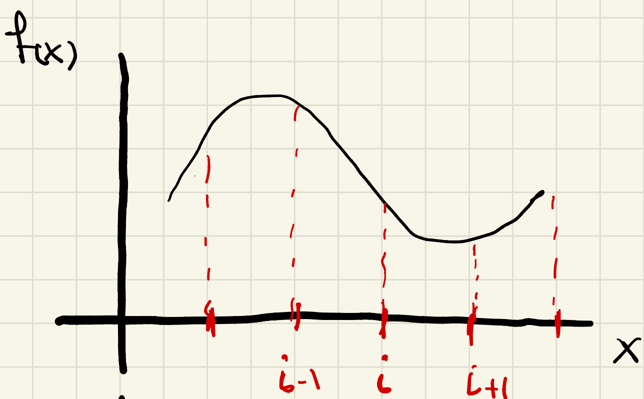
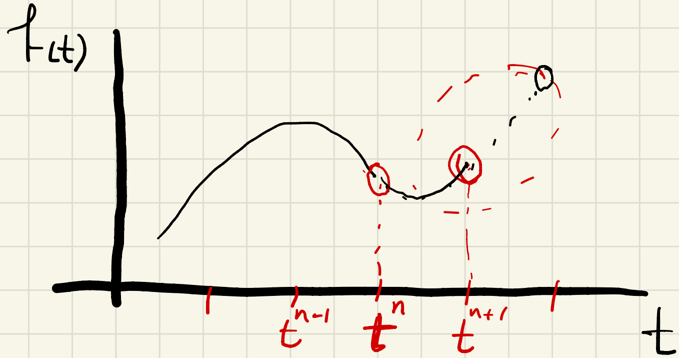
Solution of ordinary differential Equations (ODE)

- first order ODE's
- high order ODE can be converted to a system of first order ODE's
- Classes of ODE's:
  - Initial Value Problems
  - Boundary Value Problems
  - Eigenvalue Problems



$$f'_i \approx \alpha f_{(i-1)} + \beta f_i + \gamma f_{i+1} + \dots$$



$$f(t=0) \longrightarrow f(t=t_f)$$

first order ODE:

$$\text{(*)} \quad \frac{dy}{dt} = f(y,t) \quad ; \quad \underset{\text{initial Condition}}{y(0) = y_0}$$

find $y(t)$ for $0 < t \le t_f$

Numerically: $t^{n+1} = t^{n} + \underline{\Delta t} \rightarrow 0 \le t \le t^{N}$

$$\downarrow$$

$$y^{n+1} = y(t^{n+1})$$

$$\downarrow$$

$$y^{n+2}$$

$$\downarrow \text{ until}$$

$$t_f$$

Taylor Series Method:

Expand the Solution @ t^{n+1} about the Solution @ t^n $(\Delta t = h)$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \cdots$$

from $\textcircled{*}$ $\longrightarrow$ $\underline{y_n' = f(y_n, t_n)}$

if I stop @ $\swarrow$ $\Rightarrow$ Second order Method (locally)

to approximate higher Order derivatives:

$$y'' = \frac{dy'}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{f} = \underline{f_t} + \underline{f f_y}$$

Impractical $\Rightarrow$ first two term

Euler Method:

$$y^{n+1} = y^n + h f(y^n, t^n)$$

Q: how does it work?

1. Start with initial Condition: t_0
2. March forward $y_1 \rightarrow y_2 \rightarrow \cdots y_n$

from Taylor Series Expansion : - 2^{nd} order
One time Step
- 1^{st} order
global error to t^f

Class of time Integrators :

$$y_{n+1} = y_n + h f(y_n + t_n)$$

* Multi-step

One-Step Method

* Explicit / Implicit

Explicit Scheme

- Convergence ⟶ accuracy
- NEW Concept in time Ingeration

Numerical Stability :

* Critical property differential equations
* Exact Solution is well behaved
⟹ never grows unbounded

* Numerical Solution grows unbounded
⟹ Numerical Instability

Given: $y' = f(y,t)$

$\Rightarrow$ all dependant on the step size: $(h, \Delta t)$

Stable numerical schemes:

Solution will not grow unbounded (blow-up) with any choice of 'h'

Unstable numerical scheme:

Solution blows-up with any choice of "h"

$\Rightarrow$ $\boxed{\text{does not depend on accuracy}}$

Conditionally Stable numerical scheme:

Certain Choice of h leads to stability

Stability Analysis:

Determine the stability Properly

* Perform the analysis for $\otimes$ assuming it has features of the general Eqn.
* Perform two-dimensional Taylor Series Expansion

$$f(y,t) = \overset{\alpha_1}{f(y_0,t_0)} + (t-t_0)\overset{\alpha_2}{\frac{\partial f}{\partial t}(y_0,t_0)} + (y-y_0)\overset{\lambda = \text{const}}{\frac{\partial f}{\partial y}(y_0,t_0)}$$
$$+ \frac{1}{2!}\left[(t-t_0)^2\frac{\partial^2 f}{\partial t^2} + 2(t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\,\partial y} + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}\right] + \cdots$$

Collecting linear terms: $y' = \lambda y + \alpha_1 + \alpha_2 t + \cdots$

$y' = \lambda y$ ⟶ "model Problem"

↳ Exponential Soln & Worst case

It turns out ⟹ inhomogeneous terms do not significantly affect the results of stability analysis

λ ⟶ complex $\lambda_R + i\lambda_I$

Stability Analysis for Explicit Euler (EE)

$$y^{n+1} = y^n + h f(y^n, t^n)$$

$$\xrightarrow[\text{Problem}]{\text{model}} \quad y^{n+1} = y^n + h\lambda y^n$$

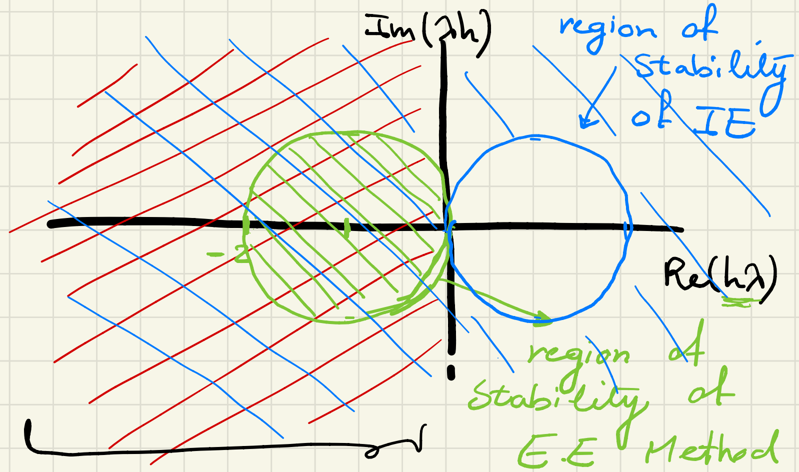
$$= y^n (1 + \lambda h)$$

for complex λ, we have

$$y^n = y^0 (1 + \lambda h)^n = y^0 \sigma^n$$

amplification factor

$$y' = \lambda y \longrightarrow e^{\lambda t} \xrightarrow{\text{exact}} \lambda_R < 0$$



Cond. Sable scheme $\Longrightarrow$ $|\sigma| \leq 1$

$$\underbrace{(1+\lambda_R h)^2 + \lambda_I^2 h^2}_{\text{circle}} = 1$$

for purely (λ_R) real : $h \leq \dfrac{2}{|\lambda_R|}$

for unstable soln :

$$\underbrace{|1+\lambda h|}_{\substack{\text{negative with magnitude} \\ \text{greater than } 1}} > 1$$

$\Rightarrow y^n = (1+\lambda h)^n y_c$

$\Longrightarrow$ Oscillations with chage of sign at every time step

Implicit or Backward Euler:

* One-step
* Implicit scheme

$$y^{n+1} = y^{n} + hf(y^{n+1}, t^{n+1})$$

if f nonlinear: solve a nonlinear algebraic Eqn.

$\Rightarrow$ iteratively

$\Rightarrow$ higher cost than Exp.

$\Rightarrow$ Much better stability

Stability analysis: (model Problem)

$$y^{n+1} = y^{n} + \lambda h y^{n+1}$$

$\rightarrow$ solve for y^{n+1}

$$y^{n+1} = \underbrace{(1-\lambda h)^{-1}}_{\sigma} y^{n} \Rightarrow y^{n+1} = \sigma^{n} y_0$$

$$\sigma = \frac{1}{1-\lambda h}$$

$$\sigma = \frac{1}{(1-\lambda_R h) + i\lambda_I h} \xrightarrow[\text{and Phase factor}]{\text{modulus}} \sigma = \frac{1}{Ae^{i\theta}}$$

$$\left.\begin{aligned} A &= \sqrt{(1-\lambda_R h)^2 + \lambda_I^2 h^2} \\ \theta &= -\tan^{-1}\frac{\lambda_I h}{1-\lambda_R h} \end{aligned}\right\}$$

For Stability: $|\sigma| \leq 1$

$$\frac{|e^{-i\theta}|}{A} = \frac{1}{A} \leq 1$$

For Stable exact Soln. λ_R is negative.

$$\Rightarrow A > 1 \Rightarrow \frac{1}{A} \leq 1 \Rightarrow \text{unconditionaly Stable}$$

Stability $\neq$ accuracy

Numerical Accuracy:

take the model problem:

$$y' = \lambda y$$

$$y^n = (\sigma)^n y_0$$

Exact solution: $y(t) = y_0 e^{\lambda t} = y_0^{\lambda n h} = y_0 (e^{\lambda h})^n$

determine accuracy by comparing to the exact soln.

Exact : $e^{\lambda h} = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \cdots$

E.E : $\sigma = 1 + \lambda h$

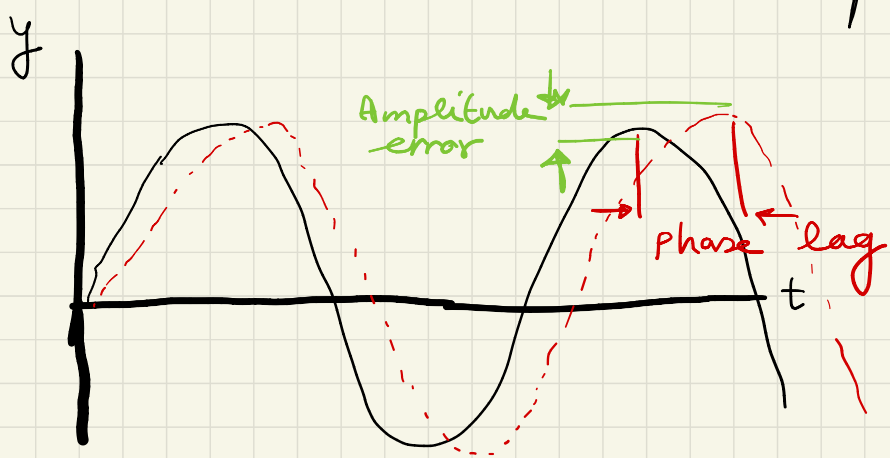
B.E : $\sigma = 1 + \lambda h + \lambda^2 h^2 + \lambda^3 h^3 + \cdots$

$\frac{1}{1 - \lambda h}$

Both methods reproduce only up to "λh"

$\Longrightarrow$ 2nd order locally
1st order globally

$\Longrightarrow$ linear analysis $\rightarrow$ upper limit
$\rightarrow$ lower order for nonlinear eqn's



for oscillatory soln. $\left\{\begin{array}{l} * \text{accuracy not informative} \\ \left.\begin{array}{l} * \text{ phase} \\ * \text{ amplitude} \end{array}\right\} \lambda_I \end{array}\right.$

$y' = i\omega y \quad , \quad y(0) = 1$

exact soln. $\quad e^{i\omega t}$, which is oscillatory

$\left\{\begin{array}{l} \text{frequency}: \ \omega \\ \text{Amplitude}: \ 1 \end{array}\right.$

Num Soln:

EE $\qquad y^n = \sigma^n y_0 \quad , \quad \sigma = 1 + i\omega h$

Amplitude: $|\sigma| = \sqrt{1 + \omega^2 h^2} \geq 1$

$\Longrightarrow$ EE unstable for purely Imag. λ

phase : $\quad \sigma = |\sigma| e^{i\theta}$

$$\theta = \tan^{-1} \omega h = \tan^{-1} \frac{\text{Im}(\sigma)}{\text{Re}(\sigma)}$$

measure of the phase error (PE) :

$$PE = \omega h - \theta = \omega h - \tan^{-1} \omega h$$

$$\tan^{-1}(\omega h) = \omega h - \frac{(\omega h)^3}{3} + \frac{(\omega h)^5}{5} + \cdots$$

$$PE = \frac{(\omega h)^3}{3} \quad \text{in one time-step}$$

$$n(PE) \Leftarrow \quad n \text{ time steps}$$

Trapezoidal Method:

$$y(t) = y_n + \int_{t^n}^{t} f(y, t')\, dt'$$

@ $t = t^{n+1}$

$$y^{n+1} = y_n + \int_{t^n}^{t^{n+1}} f(y, t)\, dt'$$

appraximate the integral with trapezoidal rule:

$$y^{n+1} = y_n + \frac{h}{2}\left[f(y^{n+1}, t^{n+1}) + f(y^n, t^n)\right]$$

* Implicit scheme
* Crank-Nicolson Scheme

A second order Equation:

$$y'' + \omega^2 y = 0 \qquad t > 0$$

$$y(0) = y_0 \quad , \quad y'(0) = 0$$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\omega^2 y_1 \end{aligned} \xrightarrow[\text{form}]{\text{in matrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$z_1' = i\omega z_1 \quad , \quad z_2' = -i\omega z_2$$

$\Rightarrow$ eigenvalues $\Rightarrow$ Both imaginary

$$Y' = AY$$

EE:

$$Y^{n+1} = Y^n + hAY^n = (I + hA)Y^n$$

IE

$$Y^{n+1} = Y^n + hAY^{n+1}$$

$$\underbrace{[I - hA]}_{B} Y^{n+1} = Y^n \Rightarrow AX = b$$

$$BY^{n+1} = Y^n$$

Model Problem : ($y' = \lambda y$)

$$y_{n+1} - y_n = \frac{h}{2}\left[\lambda y_{n+1} + \lambda y_n\right]$$

$$y_{n+1} = \underbrace{\frac{1 + \lambda h/2}{1 - \lambda h/2}}_{\sigma} y_n$$

$$\sigma = \frac{1 + \lambda h/2}{1 - \lambda h/2} = \underbrace{1 + \lambda h + \frac{\lambda^2 h^2}{2}} + \frac{\lambda^3 h^3}{\underline{\underline{4}}} + \cdots$$

Q: What is the order of accuracy?

2nd order

$$\lambda = \lambda_R + i \lambda_I$$

$$\rightarrow \sigma = \frac{1 + \lambda_R h/2 + i \lambda_I h/2}{1 - \lambda_R h/2 - i \lambda_R h/2} = \frac{A e^{i\theta}}{B e^{i\alpha}}$$

$$\sigma = \frac{A}{B} e^{i(\theta - \alpha)}$$

$$|\sigma| = \frac{A}{B}$$

for $\underline{\underline{\lambda_R < 0}}$ (exact solution bounded)

$\Rightarrow \quad A < B \Rightarrow \underline{\underline{|\sigma|}} < 1$

$\Rightarrow$ unconditionally Stable Implicit Scheme

for $\lambda = i\omega \Rightarrow A = B \Rightarrow |\sigma| = 1$

$\Rightarrow$ $\boxed{\text{no amplitude error}}$

$$PE = \omega h - 2\tan^{-1}\left(\frac{\omega h}{2}\right) = \frac{\omega h^3}{12} + \cdots$$

$\hookrightarrow$ 4 times better!
EE

Apply to TD:

$$Y^{n+1} - Y^{n} = \frac{h}{2}\left[A Y^{n+1} + A Y^{n}\right]$$

$$\underbrace{\left[I - \frac{h}{2} A\right]}\, Y^{n+1} = \underbrace{\left[I + \frac{h}{2} A\right] Y^{n}}$$

$$A \, Y^{n+1} = b$$

Runge-Kutta Methods:

→ more information can be added by including terms

RK Methods → - points between t^n & t^{n+1}
- evaluate f at intermediate points
- higher cost per time-step
- higher accuracy
- better stability properties

General form of (two-stage) 2nd Order RK formula for solving: $y' = f(y,t)$

Soln. at t^{n+1} is:

① $$y^{n+1} = y^n + \underline{\underline{\gamma_1}} k_1 + \underline{\underline{\gamma_2}} k_2$$

→ constants to be determined

k_1 & k_2:

$$k_1 = h f(y_n, t_n)$$

⇒ ensure highest accuracy

$$k_2 = h f(y_n + \underline{\underline{\beta}} k_1,\ t_n + \underline{\underline{\alpha}} h)$$

Taylor Series expansion of $y(t^{n+1})$:

$$y^{n+1} = y^n + h y_n' + \frac{h^2}{2} y_n'' + \cdots$$

$$y_n' = f(y_n, t_n)$$

using chain rule :

$$y'' = f_t + f f_y$$ → partial derivatives of f w.r.t. t & y

⟹ (2)

$$y_{n+1} = y_n + h f(y_n, t_n) + \frac{h^2}{2}\left(f_{t^n} + f_n f_{y^n}\right) + \cdots$$

to establish the order of accuracy of RK Method (1) ⟹ comparing its estimate for y^{n+1} (2)

two-dimensional Taylor series expansion of k_2 leads to

$$k_2 = h\left[f(y_n, t_n) + \beta k_1 f_{y_n} + \alpha h f_{t_n} + O(h^2)\right]$$

noting that $k_1 = h f(y_n, t_n)$, substituting in (1)

$$y^{n+1} = y^n + (\gamma_1+\gamma_2) h f_n + \gamma_2 \beta h^2 f_n f_{y_n} + \gamma_2 \alpha h^2 f_{t_n} + \cdots$$

matching similar coefficients:

$$\left.\begin{aligned} \gamma_1 + \gamma_2 &= 1 \\ \gamma_2 \alpha &= 1/2 \\ \gamma_2 \beta &= 1/2 \end{aligned}\right\}$$

3 equations

4 unknowns

$\Rightarrow$ α = free parameter

$$\Rightarrow \gamma_2 = \frac{1}{2\alpha}, \quad \beta = \alpha, \quad \gamma_1 = 1 - \frac{1}{2\alpha}$$

$\Rightarrow$ One-parameter family of 2nd order RK formulas:

$$k_1 = h f(y_n, t_n)$$

$$k_2 = h f(y_n + \alpha k_1, t_n + \alpha h)$$

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2$$

usually: $\alpha = 1/2$

predicted value $\leftarrow$ $y^*_{n+1/2} = y_n + \frac{h}{2} f(y_n, t_n)$

corrected value $\leftarrow$ $y_{n+1} = y_n + h f(y^*_{n+1/2}, t_{n+1/2})$

Stability analysis:

model eqn. $y' = \lambda y$, substitute in (1)

$\Longrightarrow$ $k_1 = \lambda h y_n$

$k_2 = h(\lambda y_n + \alpha h \lambda^2 h y_n) = \lambda h (1 + h\lambda) y_n$

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right)\lambda h y_n + \frac{1}{2\alpha}\lambda h (1 + \alpha \lambda h) y_n$$

$$= y_n \underbrace{\left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right)}_{\sigma}$$

$\Longrightarrow$ 2nd order

$\Longrightarrow$ for stability we must have

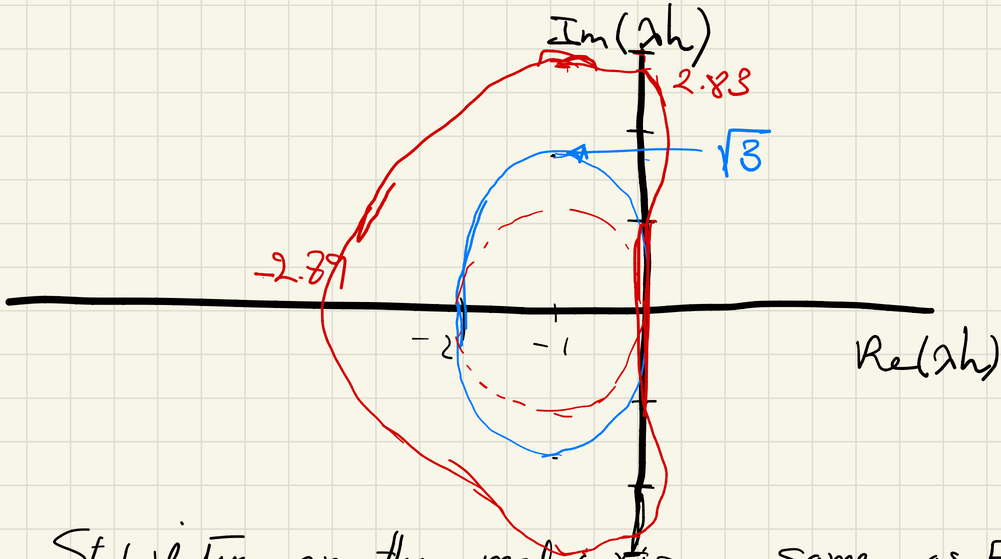
$|\sigma| \leq 1$

to find the stability region

$\Longrightarrow$ $\sigma = e^{i\theta} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right)$

$\downarrow$

find complex roots λh
for different values of θ



- Stability on the real axis = same as EE
- Significant improvement for complex λ
- for purely imaginary λ : $\boxed{\lambda = i\omega}$

unstable $\Leftarrow$ $|\sigma| = \sqrt{1 + \frac{\omega^4 h^2}{4}} > 0$

for small values of ωh $\Rightarrow$ less unstable than EE

Example : Amplification factor

consider $y' = i\omega y$ $\quad y(0) = 1$

Use EE & RK2 schemes

Integration for 100 time-steps $\omega h = 0.2$

$\Rightarrow$ $t = 0 \longrightarrow 20/\omega$

each numerical soln. after 100 iterations,

$$y = \underset{\substack{\downarrow \\ \text{amplification factor}}}{\sigma}^{100} y_0$$

EE : $|\sigma| = \sqrt{1 + \omega^2 h^2} = 1.0198 \rightarrow |y| = 7.10$

RK2 : $|\sigma| = 1.0002 \rightarrow \underline{|y| = 1.02}$

$\downarrow$ 2%.

Phase Error :

real & Imaginary parts of σ for $\lambda = i\omega$

$$PE = \omega h - \tan^{-1}\left(\frac{\omega h}{1 - \frac{\omega^2 h^2}{2}}\right)$$

$$PE = - \frac{\omega^3 h^3}{6} + \dots .$$

only factor of 2 better than EE, but opposite sign

$\Rightarrow$ negative phase error $\Rightarrow$ phase lead

The most widely used RK Method is the 4th order formula:

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}(k_2+k_3) + \frac{1}{6}k_4$$

where,

$$k_1 = hf(y_n, t_n)$$

$$k_2 = hf(y_n + \tfrac{1}{2}k_1, t_n + \tfrac{h}{2})$$

$$k_3 = hf(y_n + \tfrac{1}{2}k_2, t_n + \tfrac{h}{2})$$

$$k_4 = hf(y_n + k_3, t_n + h)$$

Stability Analysis: (model eqn. $y' = \lambda y$)

$$y_{n+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right) y_n$$

$\Rightarrow$ fourth Order accurate

$\Rightarrow$ for Plotting the stability diagram find roots of the following 4^{th} order polynomial with complex coefficients:

$$\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \lambda^4 h^4/24 + 1 - e^{i\theta} = 0 \quad \text{for}$$

$$0 \leq \theta \leq \pi$$

⇒ Improvement over the stability region of RK2 ⇒ larger time-steps
faster time to Solution

⇒ large Stability region of Imaginary axis

⇒ Stability region in Positive $Re(\lambda)$
⇒ artificially stable

Multi-step Methods :

higher order accuracy achieved by using data from prior to $t_n \to t_{n-1}, t_{n-2}$

→ Multi-step Methods
→ higher computer memory
→ not self starting → usually another method (EE, for example) start the calculation.

Leap-frog method

$$\boxed{f_i' = \frac{f_{i+1} - f_{i-1}}{2h}}$$

$$y_{n+1} = y_{n-1} + 2h f(y_n, t_n) + O(h^3)$$

$\rightarrow$ Central difference formula for y'

$\Longrightarrow$ 2nd Order

$\Longrightarrow$ y_0 (initial condition)

$\Longrightarrow$ y_1 (EE)

<u>Stability Analysis :</u> $(y' = \lambda y)$

$$y_{n+1} - y_{n-1} = 2\lambda h y_n$$

to solve it, a solution of the form

$$y_n = \sigma^n y_0$$

in $(*)$ leads to

$$\sigma^{n+1} - \sigma^{n-1} = 2h\lambda\sigma^n$$

dividing by σ^{n-1}, we get a quadratic eqn. for σ :

$$\sigma^2 - 2h\lambda\sigma - 1 = 0$$

$$\Longrightarrow \sigma_{1,2} = \lambda h \pm \sqrt{h^2\lambda^2 + 1}$$

having more than one root is the characteristic of multi-step methods :

$$\sigma_1 = \lambda h + \sqrt{\lambda^2 h^2 + 1} = 1 + \underbrace{\lambda h + \frac{\lambda^2 h^2}{2}}_{\text{2nd order accurate}} - \frac{\lambda^4 h^4}{8} + \cdots$$

$$\sigma_2 = \lambda h - \sqrt{\lambda^2 h^2 + 1} = -1 + \lambda h - \tfrac{1}{2}\lambda^2 h^2 + \cdots$$

spurious ⟹ source of numerical problems.

- $h = 0 \Rightarrow$ spurious root $\neq 1$
- for λ real and negative

  $\Rightarrow |\sigma_2| > 1 \Rightarrow$ leads to instability

- linear problem : solution is a linear combination of σ_1, σ_2

$$y_n = c_1 \sigma_1^n + c_2 \sigma_2^n$$

c_1, c_2 → starting conditions y_0 & y_1

$n = 0$ & $n = 1$

$$\begin{cases} y_0 = c_1 + c_2 \\ y_1 = c_1\sigma_1 + c_2\sigma_2 \end{cases}$$

for $i\omega = \lambda \Rightarrow$ if $|\omega h| \le 1 \Rightarrow |\sigma_{1,2}| = 1$

$\Rightarrow$ no amplitude errors

if $|\omega h| > 1 \Rightarrow |\sigma_{1,2}| = \left|\omega h \pm \sqrt{\omega^2 h^2 - 1}\right|$

$\Rightarrow$ unstable

Adams - Bashforth method :

Using Taylor Series Expansion :

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \cdots$$

Substitute $y_n' = f(y_n, t_n)$

A first order finite difference approximation for y_n'' :

$$y_n'' = \frac{f(y_n, t_n) - f(y_{n-1}, t_{n-1})}{h} + O(h)$$

$\Rightarrow$

$$y_{n+1} = y_n + \frac{3h}{2} f(y_n, t_n) - h/2\, f(y_{n-1}, t_{n-1}) + O(h^3)$$

$\hookrightarrow$ globally 2nd Order

Stability Analysis:

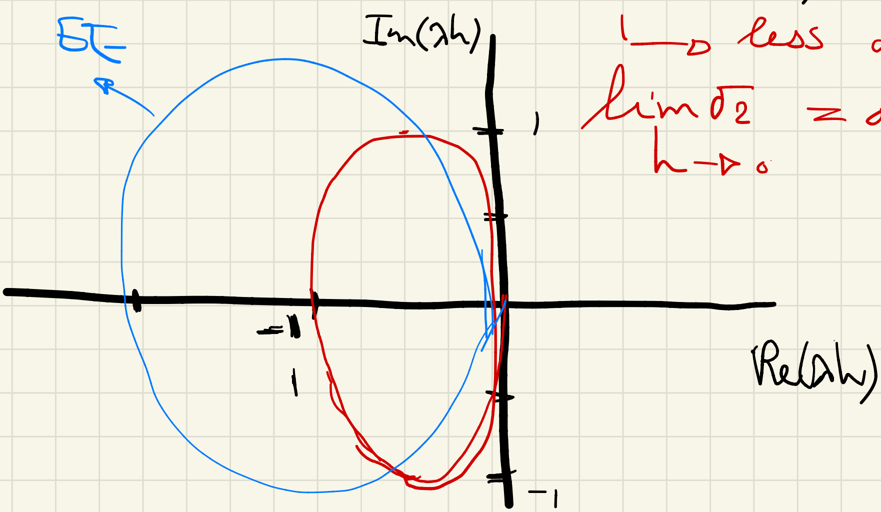
$$y_{n+1} - \left(1 + 3\lambda h/2\right) y_n + \frac{\lambda h}{2} y_{n-1} = 0$$

$\Rightarrow$ Quadratic eqn. for σ:

$$\sigma_{1,2} = \frac{1}{2}\left[1 + \frac{3}{2}\lambda h \pm \sqrt{1 + \lambda h + \frac{9}{4}\lambda^2 h^2}\right]$$

$$\sigma_1 = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + O(h^3)$$

$$\sigma_2 = \frac{1}{2}\lambda h - \frac{1}{2}\lambda^2 h^2 + O(h^3)$$



$\hookrightarrow$ less dangerous

$$\lim_{h \to 0} \sigma_2 = 0$$

- more limiting than EE & RK2
- unstable for pure imaginary λs
- very mild instability
- for the problem at $y' = -i\omega y$

$$|\sigma_1|^{100} = 1.04 \qquad 4\% \text{ error}$$

slightly worse than RK2!

TD:

$$y_{n+1} = y_n + \frac{3h}{2} f(y_n, t_n) - \frac{h}{2} f(y_{n-1}, t_{n-1})$$

Example $\quad \underline{Y}' = \underbrace{A}\, \underline{Y}$

$$\underline{Y}^{n+1} = \underbrace{\underline{Y}^{n} + \frac{3h}{2} A \underline{Y}^{n}} - \frac{h}{2} A \underline{Y}^{n-1}$$

$$\underline{Y}^{n+1}_{\ 1} = \underbrace{\left[I + \frac{3h}{2} A\right] \underline{Y}_1^{n}}_{a} - \frac{h}{2} \underbrace{A \underline{Y}_1^{n-1}}_{a}$$

$\underline{\ 0\ }$

EE (first step) $\quad \underline{Y}^{1} = \underline{Y}^{0} + hA\underline{Y}^{0} = \underbrace{[I + hA]\underline{Y}^{0}}_{a}$

System of first order differential Eqn's :

- higher order ODE can be converted to a system of 1st order ODE's
  - → Chemical reations
  - → Structure Vibration

- In generic form

$$y' = f(y,t) \; ; \quad y(0) = y_0$$

→ y is a vector with elements y_i

→ $f(y_1, y_2, \dots y_m, t)$ is a vector function with elements

$$f_i(y_1, y_2, \dots y_m, t), \quad i = 1, 2, \dots, m$$

Application of time integration schemes are straight-forward

EG 1

$$y_i^{n+1} = y_i^n + h f_i(y_1^n, y_2^n, \dots y_m^n, t_n)$$

$$i = 1, 2, 3, \dots, m$$

There is only one difference with one ODE.
Stifness Property

lets discuss stifness in connection with linear systems:

$$\boxed{\frac{dy}{dt} = Ay}$$

$A \rightarrow$ $m \times m$ constant matrix

model problem for a system of ODE's

All eigenvalues of A are real and negative $\Longrightarrow$ bounded Soln.

EE:

$$y^{n+1} = y^n + hAy^n = (I + hA)y^n$$

$$y^n = (I + hA)^n y_0$$

$$B^n = (I + hA)^n \rightarrow 0$$

for large n.

$\Longrightarrow$ from linear Algebra $\Longrightarrow$ magnitude of its eigenvalues $(\alpha_i\text{'s}) < 1$

$\Rightarrow \alpha_i = 1 + h\lambda_i$

↓ eigenvalues of A

$\Rightarrow |1 + \lambda_i h| \leq 1 \Rightarrow h \leq \dfrac{2}{|\lambda|_{max}}$

if the range of the magnitudes of eigenvalues is large

$\left(\dfrac{|\lambda|_{max}}{|\lambda|_{min}} \gg 1 \right) \Rightarrow$ Stiff system

Since the step size is limited by the part of the soln with the "fastest" respons time (largest λ) $\Rightarrow$ large # of steps.

long time behavior $\Rightarrow$ small time-steps

$\longrightarrow$ implicit scheme

Numerical Soln. of Partial differential equations (PDE)s

* Solid Mechanics
- vibrations
- elasticity
- - - - .

* Acoustics (waves)

* heat & mass transfer

faster computn ⇒ numerical soln.

→ More challenging than ODE's

Semi-Discretization:

PDE → converted easily to a system of ODEs

→ finite difference approximation for derivatives in all but one of the dimensions.

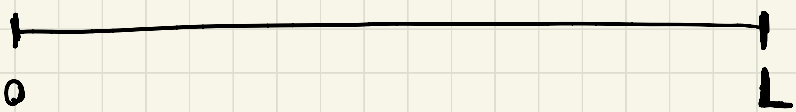
Example: One-dimensional diffusion equation (heat equation) for $\phi(x,t)$

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

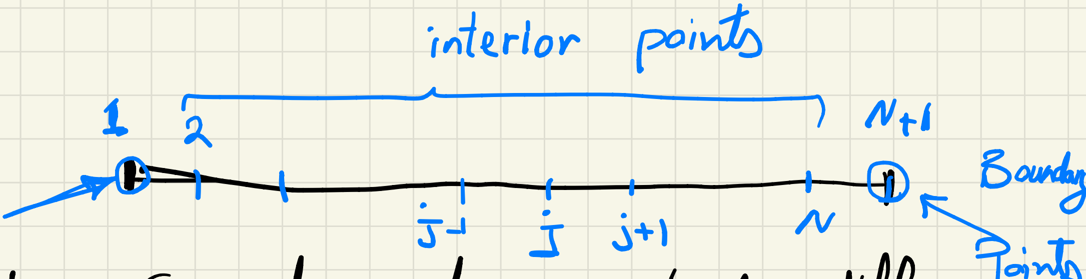
α: diffusion coefficient

Suppose the boundary and initial conditions are :

$$\varphi(0,t) = \varphi(L,t) = 0 \quad \& \quad \varphi(x,0) = g(x)$$

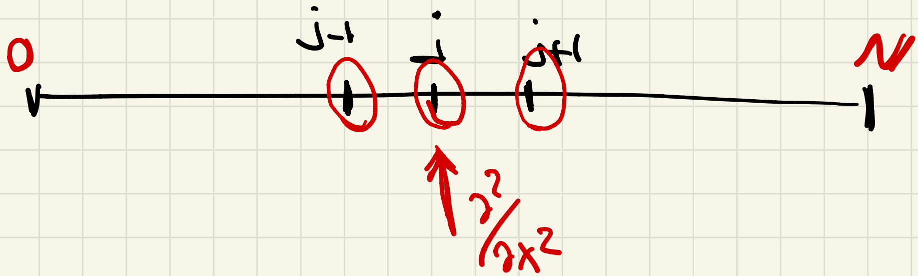


We discretize the coordinate "x" with $N+1$ uniformly spaced grid points



Use Second order central difference scheme to approximate the second derivative :

$$\frac{\partial^2 \varphi_j}{\partial x^2} = \frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{\Delta x^2}$$



$$\underbrace{\frac{\partial \varphi_j}{\partial t} = \alpha \frac{\partial^2 \varphi_j}{\partial x^2}}_{\text{PDE}} = \alpha \frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{\Delta x^2}$$

PDE $\longrightarrow$ System of ODE

where $\varphi_j = \varphi(x_j, t)$.

In matrix form

$$\frac{d}{dt}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{bmatrix} = \frac{\alpha}{\Delta x^2}\begin{bmatrix} \boxed{} \\ 1 \quad -2 \quad 1 \quad - \quad \text{---} \\ 0 \quad 1 \quad -2 \quad 1 \quad \text{---} \\ \vdots \\ \cdots \quad 1 \quad -2 \quad 1 \\ \boxed{} \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{N-2} \\ \varphi_{N-1} \\ \varphi_N \end{bmatrix}$$

$$\frac{d}{dt}(\varphi_j) = \alpha\,\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{\Delta x^2}$$

$$j=2 \longrightarrow \frac{d}{dt}[\varphi_2] = \frac{\alpha}{\Delta x^2}(\varphi_3 - 2\varphi_2 + \varphi_1)$$

$$j=N-1 \longrightarrow d/dt[\varphi_{N-1}] = \frac{\alpha}{\Delta x^2}(\varphi_N - 2\varphi_{N-1} + \varphi_{N-2}]$$

$$j=1 \longrightarrow d/dt[\varphi_1] = \frac{\alpha}{\Delta x^2}(\varphi_2 - 2\varphi_1 + \varphi_0)$$

$$j=N \longrightarrow d/dt[\varphi_N] = \frac{\alpha}{\Delta x^2}(\varphi_{N+1} - 2\varphi_N + \varphi_{N-1})$$

$$\frac{d}{dt}\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{bmatrix} = \frac{\alpha}{\Delta x^2}\underbrace{\begin{bmatrix}-2 & 1 & \cdots & & \\ 1 & -2 & 1 & \cdots & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ & & & 1 & -2\end{bmatrix}}_{A}\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_N\end{bmatrix} + \frac{\alpha}{\Delta x^2}\underbrace{\begin{bmatrix}\varphi_0\\ \vdots\\ \varphi_{N+1}\end{bmatrix}}_{\vec{b}}$$

BC

$$\varphi_0 = \varphi$$

$$\varphi_{N+1} = 0$$

$$\vec{\varphi} = [\varphi_1, \varphi_2, \cdots, \varphi_N]$$

$$\frac{d}{dt}\vec{\varphi} = \underbrace{\frac{\alpha}{\Delta x^2} A}_{B}\vec{\varphi} + \vec{b}$$

$$\boxed{\frac{d}{dt}\vec{\varphi} = B\vec{\varphi} + \vec{b}}$$

A is a banded matrix, compact notation

$$B = \frac{\alpha}{\Delta x^2} A\left[1, -2, 1\right]$$

lower diagonal   diagonal   upper diagonal

ODE system solve using any of the numerical methods introduced ODES

$\Rightarrow$ RK Methods

$\Rightarrow$ Multi-step methods

$\rightarrow$ consider the notion of <u>stiffness</u>

$$A^{(n-1)\times(n-1)} \xrightarrow[\text{eigenvalues}]{\#} (n-1)$$

Range of eigenvalues of A determine whether the system is stiff or not.

$\rightarrow$ Analytical expression are not always available

→ for A eigenvalues are

$$\lambda_j = \frac{\alpha}{\Delta x^2} \left(-2 + 2\cos \frac{\pi j}{N} \right)$$

$$j = 1, 2, \dots, N-1$$

The eigenvalue with smallest magnitude is :

$$\lambda_1 = \frac{\alpha}{\Delta x^2} \left(-2 + 2\cos \frac{\pi}{N} \right)$$

for large N :

$$\cos \frac{\pi}{N} = 1 - \frac{1}{2!} \left(\frac{\pi}{N} \right)^2 + \frac{1}{4!} \left(\frac{\pi}{N} \right)^4 - \dots$$

converges rapidly : Retain the first 2. terms

$$\lambda_1 \approx -\frac{\pi^2 \alpha}{N^2 \Delta x^2}$$

& for large N we get :

$$\lambda_{N-1} \approx -4\alpha / \Delta x^2$$

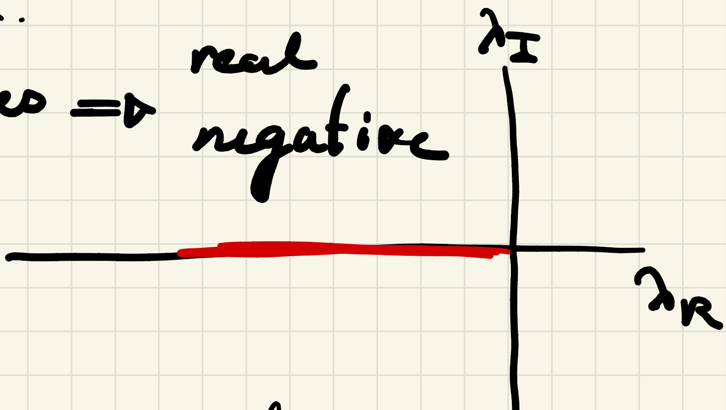
Therefore, the ratio of the eigenvalues:

$$\left|\frac{\lambda_{N-1}}{\lambda_1}\right| \approx \frac{4N^2}{\pi^2} \sim N^2$$

$$\boxed{N = 20} \longrightarrow N^2 = 400$$

λs insight into the physical behavior of the soln.

eigenvalues $\Rightarrow$ real negative



to see the role of eigenvalues:

Diagonalize A (linear Algebra)

$$A = S \Lambda S^{-1}$$

$$\Lambda = S^{-1} A S$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$$

$\Lambda \rightarrow$ eigenvalues

$\leftarrow S \rightarrow$ eigenvectors

Substitute A with $S\Lambda S^{-1}$

in $\frac{\partial \varphi}{\partial t} = A\varphi$

$$S^{-1}\frac{d\varphi}{dt} = \underbrace{S^{-1}S}_{I}\Lambda S^{-1}\varphi \longrightarrow \frac{d}{dt}\underbrace{S^{-1}\varphi}_{\psi} = \Lambda \underbrace{S^{-1}\varphi}_{\psi}$$

$$\frac{d}{dt}\psi = \Lambda \psi$$

$\hookrightarrow$ diagonal $\Rightarrow$ equations are decoupled

$$\psi_j(t) = \psi_j(0)\, e^{\lambda_j t}$$

$$\varphi = S\psi = \psi_1 s^{(1)} + \psi_2 s^{(2)} + \dots \psi_{N-1} s^{(N-1)}$$

$$A = \begin{cases} \text{eigenvalues} \Rightarrow \text{Temporal behavior of The Soln.} \\ \text{eigenvectors} \Rightarrow \text{Spatial behavior of the Soln.} \end{cases}$$

$A = [1 \quad -2 \quad 1]$

negative & real eigenvalues

$\Rightarrow$ decaying soln. in time

$\Rightarrow$ Rate of decay: magnitude of the eigenvalues

Solve the system of ODEs:

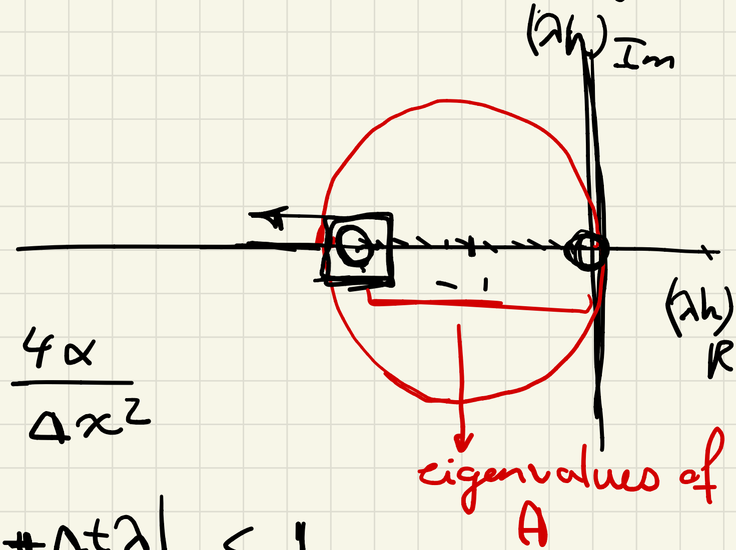
EE for time advancement

$\rightarrow$ Conditionally Stable

Q. Estimate Δt_{max} ?

$$\frac{d\phi}{dt} = \frac{\alpha}{\Delta x^2} A[1, -2, 1] \phi$$

The Stability depends on the eigenvalues of the system. having the largest mag:



$$\lambda_{N-1} \approx -\frac{4\alpha}{\Delta x^2}$$

model

Problem EE: $|1+\Delta t\lambda|_{max} \leq 1$

$$\Delta t_{max} = \frac{2}{\lambda_{max}} = \frac{\Delta x^2}{2\alpha}$$

$\alpha = 1$, $\Delta x = 0.05 \longrightarrow \Delta t_{max} = 0.00125$

$\Delta t < 0.00125$ Stable $\quad \Delta t = 0.001$

$\Delta t > 0.00125$ unstable $\quad \Delta t = 0.0015$

$$\frac{d\varphi}{dt} = \frac{\alpha}{\Delta x^2} A[1,-2,1] + \cancel{b}$$

$$\varphi_0 = 0$$

$$\varphi_N = 0$$

GE :

$$\frac{dy}{dt} = Ay \quad \text{(model Problem)}$$

$$\frac{y^{n+1} - y^n}{\Delta t} = Ay^n \rightarrow y^{n+1} = y^n + \Delta t A y^n$$

$$\Rightarrow y^{n+1} = (I^{(n-1)\times(n-1)} + \Delta t A) y^n$$

$$(T_{or}^{n+1})\ \varphi^{n+1} = \left(I + \underbrace{\frac{\Delta t\,\alpha}{\Delta x^2}}_{\sigma} A \right) \varphi^n$$

$\varphi^{n+1} \downarrow$ tstep+1

$\varphi^n \downarrow$ tstep

Von - Neuman Stability Analysis :

Matrix Stability analysis →

look at the Matrix and extract eigenvalues

→ B'C is included.

disadvantage → Diagonalize A and extract the eigenvalues ⟹ not possible always.

→ Simplifies the BC → $\boxed{\text{periodic B.C.}}$

B.C commonly do not influence Stability criteria.

→ Constant coeff differential eqn.

→ uniformly spaced grid.

Heat eqn. :

- FE time advancement
- 2nd order central difference

time counter

$$\varphi_j^{(n+1)} = \varphi_j^{(n)} + \frac{\alpha \Delta t}{\Delta x^2}\left(\varphi_{j+1}^{(n)} - 2\varphi_j^{(n)} + \varphi_{j-1}^{(n)}\right) \quad (*)$$

grid index

$\boxed{\text{Key Part}}$: Assume soln. of the form:

$$\varphi_j^{(n)} = \sigma^n e^{ikx_j}$$

$$\varphi = \sum \beta_i e^{ikx_j}$$

Substitute in $(*)$

$$\sigma^{n+1} e^{ikx_j} = \sigma^n e^{ikx_j} + \frac{\alpha \Delta t}{\Delta x^2}\sigma^n \left(e^{ikx_{j+1}} - 2e^{ikx_j} + e^{ikx_{j-1}}\right)$$

$x_{j+1} = \Delta x + x_j$

$x_{j-1} = -\Delta x + x_j$

$\xrightarrow{\text{devide}} \sigma^n e^{ikx_j}$

$$\underbrace{\sigma}_{\text{amplification factor}} = 1 + \left(\frac{\alpha \Delta t}{\Delta x^2}\right)\left[2\cos(k\Delta x) - 2\right]$$

bounded exact Soln: $|\sigma| \leq 1$

$\downarrow$ for Stability

$$\left| 1 + \left(\frac{\alpha \Delta t}{\Delta x^2}\right)\left[2\cos(k\Delta x) - 2\right] \right| \leq 1$$

$$-1 \leq 1 + \frac{\alpha \Delta t}{\Delta x^2}\left[2\cos(k\Delta x) - 2\right] \leq 1$$

always satisfied $\left[2\cos(k\Delta x) - 2\right] \leq 0$

$$\left(\frac{\alpha \Delta t}{\Delta x^2}\right)\left[2\cos(k\Delta x) - 2\right] \geq -2$$

$$\Delta t \leq \frac{\Delta x^2}{\alpha\left[1 - \cos(k\Delta x)\right]}$$

$$\boxed{\Delta t \leq \frac{\Delta x^2}{2\alpha}}$$

worst case: $\cos(k\Delta x) = -1$

→ They match.

Modifeid wavenumber analysis:

Very Similar to von-Neumann analysis.

$\longrightarrow$ more straight forward

take $\quad \dfrac{d\phi}{dt} = A\phi \quad (1)$

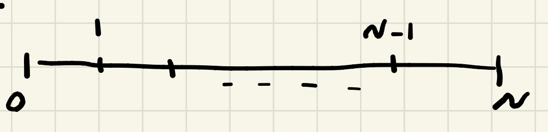
Assume a soln. $\quad \underbrace{\phi(x,t) = \psi^{(t)} e^{ikx}}_{\text{substitute in (1)}}$ $\quad$ Periodic B.C

$$\frac{d\psi}{dt} = \underbrace{-\alpha k^2}_{\text{wave-number}} \psi \qquad \equiv \qquad y' = \lambda y$$

$\longrightarrow$ in practice instead of analytical expression we use discretisation to approximate the derivative

Heat eqn. $\longrightarrow$ 2nd order central diff. $\partial^2/\partial x^2$

$$\frac{d\phi_j}{dt} = \alpha \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}, \qquad j = 1, 2, 3, \dots, N-1$$



Assume : $\underbrace{\varphi_j = \psi(t)\, e^{ik x_j}}_{\text{soln to the (semi-) discrete equation}}$

$\Rightarrow$ substituting and dividing by e^{ikx_j} :

$$\frac{d\psi}{dt} = -\frac{2\alpha}{\Delta x^2}\left[1 - \cos(k\Delta x)\right]\psi$$

or

<u>FD</u> : $\dfrac{d\psi}{dt} = -\alpha \underline{\underline{k'^2}}\, \psi(t)$

<u>exact</u> : $\dfrac{d\psi}{dt} = -\alpha \underline{\underline{k^2}}\, \psi(t)$

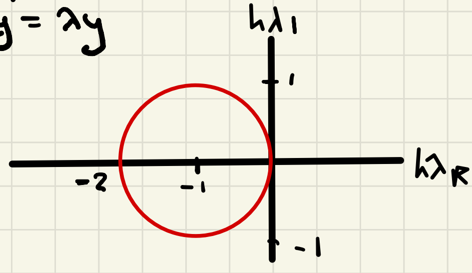
2nd FD : $k'^2 = \dfrac{+2}{\Delta x^2}\left[1 - \cos(k\Delta x)\right]$

modifeid wave-number

eqn. decoupled $\Rightarrow$ we can directly compare to the result of $y' = \lambda y$ with $\lambda = -\alpha k'^2$

Example: EG. from $y' = \lambda y$ (Heat eqn.)

$$\Delta t \leq \frac{2}{|\lambda|}$$



$$\Delta t \leq \frac{2}{\frac{2\alpha}{\Delta x^2}\left[1 - \cos(k\Delta x)\right]}$$

worst case scenario $\Rightarrow$ max limitation on Δt $\rightarrow$ $\cos(k\Delta x) = -1$

$$\Rightarrow \Delta t \leq \frac{\Delta x^2}{2\alpha} \Rightarrow \text{von-Neumann analysis.}$$

Wave equations:

Consider the following eqn:

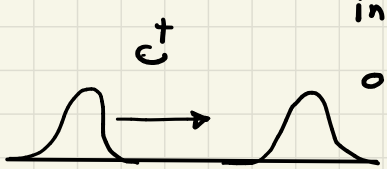
$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \qquad 0 \leq x \leq L \;,\; t \geq 0$$

wave speed

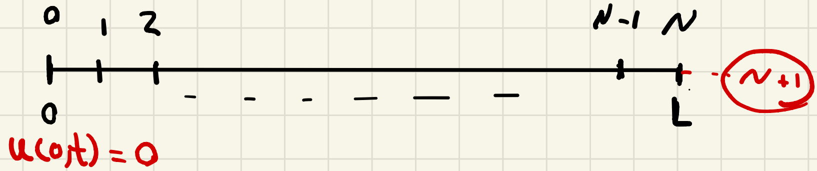
with the boundary Condition $u(0,t) = 0$

$\Rightarrow$ model equation for the convection phenomenon.

$\rightarrow$ exact soln: A wave that propagates with constant convection speed (c) in positive direction c^+ or negative direction c^-



Semi-discrete form $\Rightarrow$ discretise in space (x)



2nd order FD scheme : $\partial/\partial x$

interior points : $\partial/\partial x \, u_j = \dfrac{u_{j+1} - u_{j-1}}{2\Delta x} + O(\Delta x^2)$

$$u = [u_1, u_2, \dots, u_N]$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix} = \frac{c}{2\Delta x}\begin{bmatrix} 0 & 1 & \cdots & & & \\ -1 & 0 & 1 & 0 & \cdots & \\ 0 & -1 & 0 & 1 & \cdots & \\ & & & & & \\ & \cdots & & -1 & 0 & 1 \\ & & & & -2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{bmatrix} - \frac{c}{2\Delta x}\underbrace{\begin{bmatrix} u_0 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}}_{\text{b.c.}}$$

(a) $j=1$: $\dfrac{\partial}{\partial x} u_1 = \dfrac{u_2 - u_0}{2\Delta x}$ BC

(a) $j=N$: $\dfrac{\partial}{\partial x} u_N \overset{?}{=} \dfrac{\boxed{u_{N+1}} - u_{N-1}}{2\Delta x}$

$\partial / \partial x \longrightarrow$ 2nd FD $\times$ apply to N

$$u' = \frac{\partial u_N}{\partial x} = \alpha u_N + \beta u_{N-1}$$

$$u'_N - \alpha u_N - \beta u_{N-1} = 0$$

	u_N	u_N'	u_N''	u_N'''
u_N'	0	1	0	0
$-\alpha u_N$	$-\alpha$	0	0	0
$-\beta u_{N-1}$	$-\beta$	$+\beta \Delta x$	$-\beta \frac{\Delta x^2}{2}$	
$-\gamma u_{N-2}$	$-\gamma$	$+\gamma(2\Delta x)$	$-\gamma \frac{(2\Delta x)^2}{2}$	$- \dots$
0				

3 eqn. 3 unknowns $\longrightarrow$ 2nd order

Taylor Series expansion

$$-\beta u_{N-1} = -\beta u_N + \beta \Delta x\, u_N' - \beta \frac{\Delta x^2}{2} u_N'' + \dots$$

2 eqn. 2 unknowns
$$\begin{cases} -\alpha - \beta = 0 \\ +\beta \Delta x + 1 = 0 \end{cases} \Longrightarrow \quad \alpha = -\beta = +\frac{1}{\Delta x}, \quad \beta = \frac{-1}{\Delta x}$$

1st order

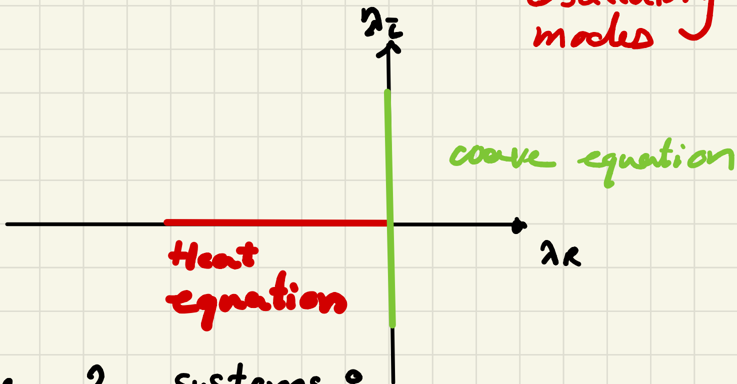
$$O(\text{scheme}) = -\beta \frac{\Delta x^2}{2} u'' = O(\Delta x)$$

$$\frac{\partial u_N}{\partial x} \simeq \frac{-u_{N-1} + u_N}{\Delta x}$$

Wave equation (eigenvalues)

$$\lambda_j = -\frac{c}{\Delta x}\left(i\cos \frac{\pi(j)}{N}\right), \quad j = 1, 2, \dots, N$$

eigenvalues are purely imaginary $\underbrace{\lambda_j = i\omega_j}_{\text{Oscillatory modes}}$



Compare the 2 systems:

examples of two limiting cases

diffusive * decaying soln. (negative real λs)

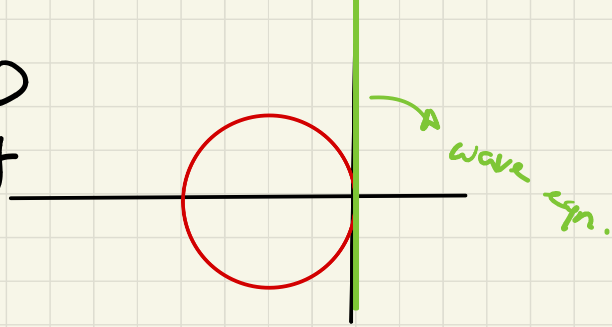
convective * Oscillatory behavior (imaginary λs)

Q: what will be the result of EE appleid to a wave eqn. above?

After 2nd order FD

⟹ purely imaginary eigenvalues

⟹ EE unstable.



EE applied to wave eqn. with 2nd order central diff:

$$\underbrace{\frac{du}{dt} = -\frac{c}{2\Delta x} B u}_{} + \cancelto{0}{\vec{b}}$$

$u_0 = a$

EE : $$\frac{y^{n+1} - y^{n}}{\Delta t} = f(y^n, t^n)$$

$$u^{n+1} = u^n - \frac{c}{2\Delta x} B u^n = \left(\underset{N\times N}{I} - \frac{c}{2\Delta x} \overset{N\times N}{B}\right) u^n$$

$c = 1$

$u_0 =$ known.

Modifeid wave number applied to wave eqn.

* Second order central difference

$\Rightarrow \quad \varphi_j = \psi(t) e^{ikx_j}$

in semi-discrete form:

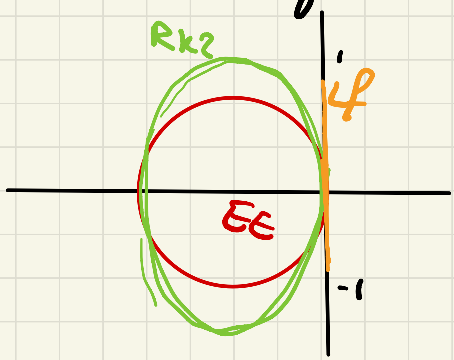
$$\frac{d\psi}{dt} = -i\underset{\sim}{k'}c\psi$$

and $$\underbrace{k' = \frac{\sin(k\Delta x)}{\Delta x}}$$

modifeid wave number for second order central diff. scheme $\longrightarrow$ $y' = \lambda y$

replace: $\lambda = -ik'c$

EE & RK2 = numerically unstable



but leap-frog $\longrightarrow$ $\Delta t = \dfrac{1}{|\lambda_i|}$

$$\Delta t_{max} = \frac{1}{k'c} = \frac{\Delta x}{c\sin(k\Delta x)}$$

The worst case scenarior :

$$\Delta t_{max} = \frac{\Delta x}{c}$$

or $\boxed{\dfrac{c\Delta t}{\Delta x}} \leq 1$

CFL number

this non-dimensional quantity is called CFL number

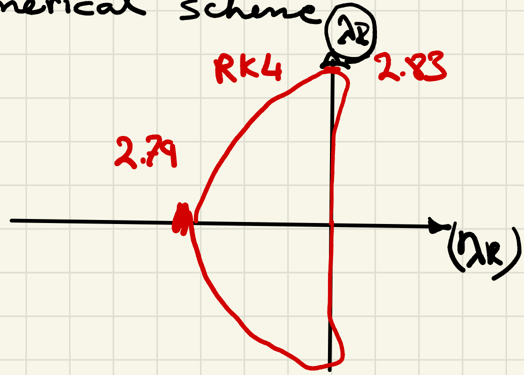
$\hookrightarrow$ Courant, Friedrich and Leay

In wave (convection-type) problems, the term "CFL number" is used as an indicator of stability of a numerical scheme.

Q : if apply RK4 with 2nd order FD

CFL ?

CFL ≤ 2.83



Leap frog : $y_{n+1} = y_{n-1} + 2h\, f(y_n, t_n)$

$$\frac{du}{dt} = Bu$$

$$u^{n+1} = u^{n-1} + 2\Delta t\, B u^n$$

$u_1 \longrightarrow$ starter scheme : EE

u_0

$$u_1 = (I + \Delta t B)\, u_0$$

$\longrightarrow$ leapfrog $\quad u^2 = u^0 + 2\Delta t B u^1$

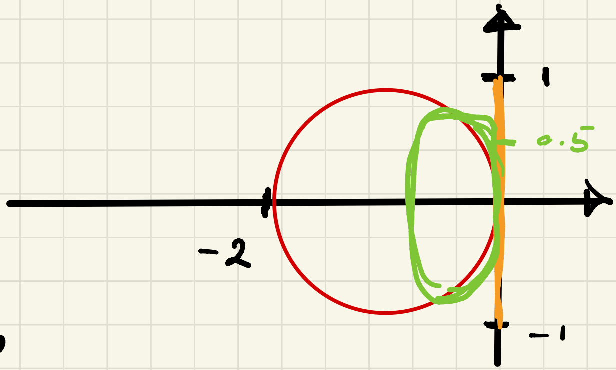
$\vdots$

Adams_Bashforth

$$y_{n+1} = y_n + \tfrac{3h}{2} f(y_n, t_n) - \tfrac{h}{2} f(y_{n-1}, t_{n-1})$$

$$u^{n+1} = u^n + \frac{3\Delta t}{2} B u^n - \frac{\Delta t}{2} B u^{n-1}$$

$$u^{n+1} = \left(I + \frac{3\Delta t}{2} B\right) u^n - \frac{\Delta t}{2} B u^{n-1}$$



CFL ?

AB

2nd order Central scheme ? $\approx \boxed{0.5}$

Example Modified wave number analysis

heat & wave eqn.

Apply modified wave number to heat-eqn.

$$\frac{d\varphi}{dt} = -\alpha k'^2 \varphi$$

for 2nd order Central diff. scheme, the worst case scenario:

$$k'^2 = \boxed{\frac{4}{\Delta x^2}}$$

We can now predict the stability of various marching methods.

EE : $\Delta t \leq \frac{\Delta x^2}{2\alpha} = \boxed{0.00125}$

$\Delta t_{max} = \frac{2}{|\lambda|}$

RK4 : $\Delta t \leq \frac{2.79\,\Delta x^2}{4\alpha} = 0.00174$

leap-frog : unstable

Similarly for convection equation:

$$\frac{d\varphi}{dt} = -ick'\varphi$$

Second order central diff the worst case

$$k' = \frac{1}{\Delta x}$$

Since ick' purely imaginary

EE : unstable

RK4 : $\Delta t \leq \frac{2.83 \Delta x}{c}$

$\Rightarrow CFL \leq 2.83$

for $\Delta x = 0.01$, $c = 1$

TD$\downarrow$

RK4 $\longrightarrow$ $0.028 = \Delta t$ $CFL \approx 2.83$

leapfrog $\longrightarrow$ $0.01 = \Delta t$ $CFL \approx 1$

AB $\longrightarrow$ $0.005 = \Delta t \sim$ $CFL \approx 0.5$

$CFL \uparrow \longrightarrow \Delta t \uparrow$

faster time to soln.